Local log-law of the wall: numerical evidences and reasons

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Abstract

Numerical studies performed with a primitive equation model on two-dimensional sinusoidal hills show that the local velocity profiles behave logarithmically to a very good approximation, from a distance from the surface of the order of the maximum hill height almost up to the top of the boundary layer. This behavior is well known for flows above homogeneous and flat topographies ("law-of-the-wall") and, more recently, investigated with respect to the large-scale ("asymptotic") averaged flows above complex topography. Furthermore, this new-found local generalized law-of-the-wall involves effective parameters showing a smooth dependence on the position along the underlying topography. This dependence is similar to the topography itself, while this property does not absolutely hold for the underlying flow, nearest to the hill surface.

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The impact of terrain features has been recognized (see, e.g., [1, 2]) to be crucial for the correct prediction of the atmosphere general circulation. This is mainly due to the key role played by the surface features in extracting momentum from the atmosphere, either through the differential pressure across the object or the vertical propagation of internal gravity waves initiated by the flow over the mountains. The problem for the weather prediction is that most of the surface disturbances are much smaller than what can be resolved by any current operational numerical model, and yet these disturbances can still have a considerable impact on the transfer of momentum from the atmosphere to the surface. For this reason the problem concerning how to parameterize the small-scale orographic effects on the large-scale dynamics attracts much attention [3, 4, 5, 6, 7].

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An important result in this direction concerns the generalization to the case of complex terrain of the well-known "law-of-the-wall" (see, e.g., [8]) valid for a neutral homogeneous turbulent boundary layer above flat terrain:

$$U(z) = \frac{u_\star}{k} \ln\left(\frac{z}{z_0}\right) \quad , \tag{1}$$

where $U(z)$ is the wind profile averaged over a time much larger than the characteristic times of turbulence, k is the Von Kàrmàn's constant, z is the distance from the surface, z_0 and $u_\star$ are the "roughness length" and the "friction velocity", respectively. Just very recently the above logarithmic law-of-the-wall for the mean velocity profile has received a rigorous analytical prove. This has been obtained in Refs. [9, 10] by combining the "rapid distortion theory" and the averaged Reynolds stress description of the mean flow.

It is probably worthwhile to recall (see, e.g., Ref. [8]) that z_0, a quantity characteristic of the surface itself, is related to the height of surface protrusions, while $u_\star$, a property of the flow, is proportional to the turbulent fluxes of momentum along the vertical (see again Ref. [8]).

It has been actually found, as a result of numerical simulations (see, e.g., Ref. [6]) and observations (both in wind tunnel, see, e.g., Refs. [11], and in nature, see, e.g., Ref. [12]), that, in a neutral homogeneous boundary layer over hilly terrain, the logarithmic shape is restored to a very reasonable approximation, from a distance from the surface of the order of the maximum hill height almost up to the top of the boundary, for $\langle U \rangle$, i.e. the velocity profile averaged over an area of linear dimensions much larger than the typical scale on which the topography varies, a regime which we would like to define as "asymptotic", in analogy with another topic we are going to discuss at the end of this Letter. More precisely, Eq. (1) still holds, with $U(z)$ replaced by $\langle U \rangle(z)$, where z is still the distance from the surface, u_* is replaced by the "effective friction velocity", $\mathsf{u}_\star^{\mathsf{eff}}$, and z_0 is replaced by the "effective roughness length", $\mathsf{z}_0^{\mathsf{eff}}$. Notice that the values of these effective parameters are considerably larger than those of the corresponding ones in the absence of any hill.

It is a matter of fact that one can be interested in the dynamics of the flow over intermediate scales between the two considered, i.e. at scales comparable with those on which the terrain varies, a regime which we like to refer to as "pre-asymptotic". For this regime, an interesting and natural question arises about the presence of some form of structural similarity and thus on the existence of logarithmic laws with effective parameters through which the dynamics can be described. If that is so, the investigation of the relation (if any) between the asymptotic and the pre-asymptotic parameters should be an interesting issue to be investigated. The above points, which are up to now largely unexplored, are the main concern of the present Letter.

To investigate the above conjectures, we have considered in detail the case-studies analyzed by Wood and Mason in Ref. [6] (hereafter, WM93 data-set), even if we have

drawn similar conclusions analyzing experiments both in a wind tunnel (see Ref. [11]) and in nature (see, e.g., Ref. [12])). In fact, the attention of Wood and Mason, as well as that of any other author involved with this topic – at least as far as we know – was focused on the areally averaged velocity field (over scales much greater than those on which topography varies) and not on the scales we referred to as the pre-asymptotic regime.

The part of WM93 data-set we have considered consists of velocity fields obtained from numerical simulations performed over three different two-dimensional topographies whose shapes are described by the following expression:

$$h(x, y) = H sin^2 \left(\frac{x\pi}{\lambda}\right) \quad . \tag{2}$$

In all modeled cases, $\lambda = 1000\ m$, while the roughness length, z_0, is 0.1 m; three different surface configurations have been considered in the present study: $H = 250\ m$ (hereafter hill H250), $H = 100\ m$ (H100) and $H = 20\ m$ (H20).

The flow is driven by a constant pressure gradient corresponding to a geostrophic wind speed of 10 m/s in the x-direction. The model was initialized with the "unperturbed" profile, i.e. the wind profile relative to a corresponding flat surface, shown as a dashed line in Fig. 1, the flattening of $U(z)$ at values of z higher than $\sim 1000\ m$ being due to the geostrophic balance occurring at the end of the Ekman layer (see, e.g., Ref. [13]) where the surface drag is vanishing.

The Wood and Mason's numerical model uses the Boussinesq approximation to the ensemble-averaged Navier–Stokes equations, together with the equation of mass conservation and a $1\frac{1}{2}$-order closure scheme through which small-scale (subgrid) motions are described. More precisely, the Reynolds' stress is expressed in the form $\tau_{ij} = \nu S_{ij}$ where $S_{ij} = (\partial_i u_j + \partial_j u_i)$ and the eddy viscosity ν is modeled as $\nu = L_m^2 S$, with $S = 1/2\ S_{ij} S_{ij}$ and L_m the mixing length, obtained by solving the equation for the turbulent kinetic energy (for major details, see Ref. [6] and the references therein).

Equations are solved by standard finite difference methods with a grid spacing of approximately 50 m. Periodic boundary conditions in both horizontal directions are also imposed. A more extensive description of the numerical model and of its evaluation against field experiments can be found in Refs. [2, 6] and in other papers of the same research group.

Our remark concerning the analysis of WM93 data-set, is that the generalized law of the wall observed in Ref. [6] for $\langle U\rangle(z)$ is certainly present, but a more intrinsic logarithmic shape

$$U(x, z) = \frac{u_\star^{\rm eff}(x)}{k} \ln\left(\frac{z}{z_0^{\rm eff}(x)}\right) \qquad \text{for} \qquad z > z_{min} \sim H \quad , \tag{3}$$

where the effective parameters now show a dependence on the horizontal position x, actually takes place at smaller scales than those focused by the authors. This fact can

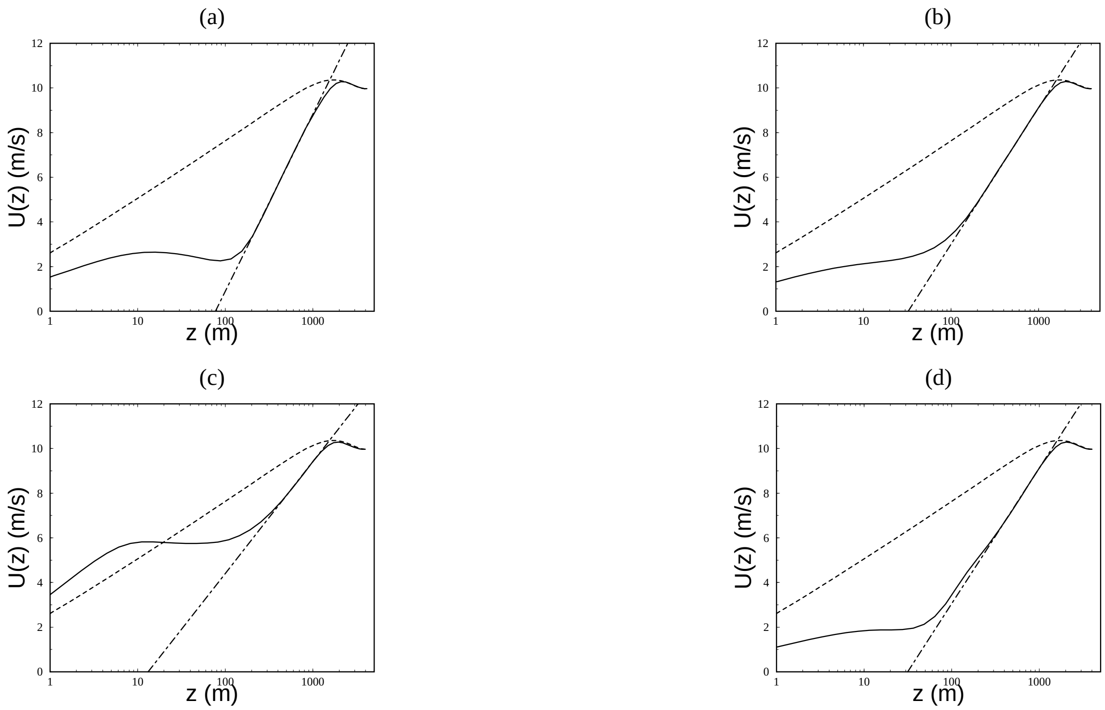

Figure 1: The local wind speed profiles $U(z) = \sqrt{u(z)^2 + v(z)^2}$ are plotted (solid lines) as a function of z for four different positions (x in Eqs. (2) and (3)) along the hill H250, corresponding to (a) $x = 0$, (b) $x = \lambda/4$, (c) $x = \lambda/2$ and (d) $x = 3\lambda/4$. No horizontal average has been performed to obtain U. The dashed lines represent the unperturbed profile. The dot-dashed lines represent the logarithmic law of Eq. (3), with parameters $u_\star^{\rm eff}(x)$ and $z_0^{\rm eff}(x)$ obtained by least-square fits performed inside the scaling regions. The values of these effective parameters are given in the text.

be easily checked from the results shown in Fig. 1, where typical behaviors for the horizontal wind speed profile (which can be thought to be representative of an area of the order of the model grid box) $U(z) = \sqrt{u(z)^2 + v(z)^2}$ (for simplicity, the dependence on the x-coordinate is omitted in the notation) as a function of z are presented in lin-log coordinates for the steepest hill H250 and for four values of the x-coordinate in Eqs. (2) and (3) corresponding to: (a) $x = 0$ (i.e. $h = 0$), (b) $x = \lambda/4$ (i.e. $h = H/2$ upwind), (c) $x = \lambda/2$ (i.e. $h = H$) and (d) $x = 3\lambda/4$ (i.e. $h = H/2$ downwind), respectively. From this figure, clean scaling regions of the type described by Eq. (3) (a straight-line with slope $u_\star^{\rm eff}(x)/k$ in these coordinates) extended up to ~ 1 decade are evident and a reliable measure of both $u_\star^{\rm eff}(x)$ and $z_0^{\rm eff}(x)$ is thus feasible by least-square fits. Specifically, for the four above positions along the hill, we have obtained the following values of $u_\star^{\rm eff}(x)$ and $z_0^{\rm eff}(x)$: 1.38 m/s, 77.3 m (for $h = 0$); 1.07 m/s, 32.4 m (for $h = H/2$ upwind); 0.87 m/s,

13.2 m (for $h = H$) and 1.05 m/s and 31.5 m (for $h = H/2$ downwind), respectively. Such values can be compared with those in the absence of any hill: $u_\star \simeq 0.44$ m/s and $z_0 \simeq 0.16$ m. Notice that the logarithmic behavior shown in the above figure is not peculiar of the four positions considered, being indeed observed for the full range of variability of x and for all the three hills.

The results of the least-square fits are summarized in Fig. 2 where both profiles $u_\star^{\text{eff}}(x)$

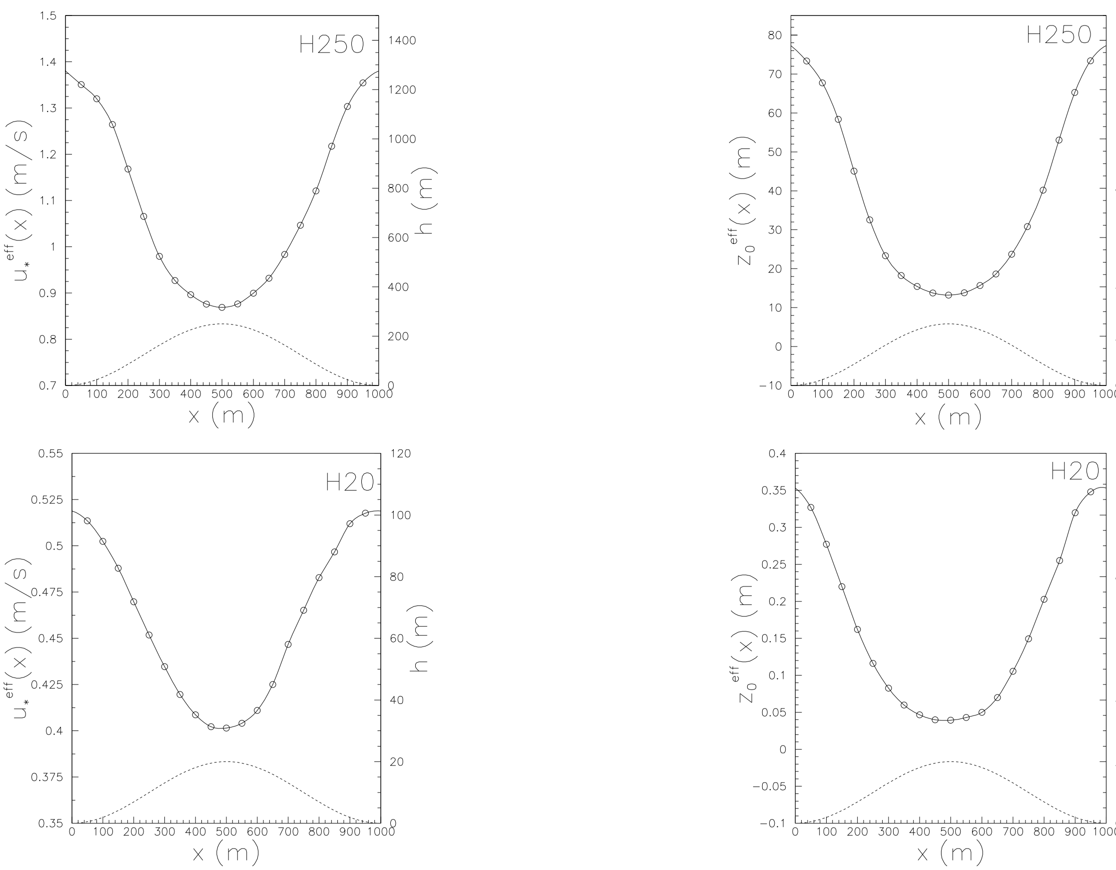

Figure 2: The measured effective parameters $u_\star^{\text{eff}}(x)$ (on the left) and $z_0^{\text{eff}}(x)$ (on the right) (represented with circles joined by a full line) as a function of the position x along the axis of the hill H250 (above) and H20 (below). The ordinate on the right of each plot is relative to the hill elevation (dashed line).

(on the left) and $z_0^{\text{eff}}(x)$ (on the right) are shown as a function of x both for the steepest hills (H250) and for the gentlest one (H20) (from above to below: notice the different scales in the ordinates both on the left and on the right hand sides). Similar behaviors have been also observed for the intermediate hill H100. Notice that both $u_\star^{\text{eff}}(x)$ and $z_0^{\text{eff}}(x)$

have a shape similar, but opposite, to that of the underlying topography, here described by Eq. (2). Furthermore, we have verified that the use of either the total horizontal speed or just the x-component of the velocity has little impact on the values of the obtained effective parameters.

Finally, we have observed that both the average values and amplitudes of these sinusoidal shapes show, to a reasonable approximation, a linear dependence on H/λ, with $z_0^{\text{eff}}(x) \mapsto z_0$ and $u_\star^{\text{eff}}(x) \mapsto u_\star$ when $H/\lambda \mapsto 0$.

Notice that the above properties are valid only for $z > z_{min} \sim H$, as already written on the right of Eq. (3), while in the underlying flow, nearest to the hill surface, the upwind-downwind asymmetry of the velocity field is more pronounced with increasing the H/λ ratio (for instance the "flow separation" occurring in the lee of steep hills and ridges, close to the surface, is a well-known feature we have observed, for instance, for the hill H250).

Some other remarks and comments are in order. The first concerns the physical meaning of dynamical quantities like $u_\star^{\text{eff}}(x)$ and $z_0^{\text{eff}}(x)$. It is natural to assume that they share the same meaning of $u_\star$ and z_0, respectively, for flat surfaces but now with a smooth dependence on the position.

The minimum of $u_\star^{\text{eff}}(x)$ occurring on the top of the hills can be easily understood, being related to curvature effects. If the streamlines are indeed curved (as it happens in our case due to the presence of the ridges), energy may be transferred between the large-scale flow and the turbulent motion (the amount of which can be measured as $u_\star^{\text{eff}\,2}(x)$). This point can be easily grasped by means of the simple argumentations presented in Ref. [14] based on the analysis of the transfer of energy in terms of the angular momentum of the flow about the axis of curvature. In this framework, it is possible to show that the flow curvature above the hill top works to transfer the energy from turbulent motion toward larger scales. This fact reduces the turbulence energy and thus $u_\star^{\text{eff}\,2}(x)$. The situation changes above the valleys, where the mechanism is reversed: energy is now released from the large-scale flow and appears as energy of the turbulent motion, and thus an enhancement of $u_\star^{\text{eff}\,2}(x)$ occurs.

Also the minimum of $z_0^{\text{eff}}(x)$ above the top of the ridge can be understood remembering [8] that, in the case of flat terrain, z_0 is related to the height of surface protrusions and thus to the size of eddies generated by the flow around them. Indeed, for hilly terrain it is reasonable to assume that the eddies placed in proximity of the surface act as a sort of effective protrusions for the above pre-asymptotic flow. In this picture, it is evident that such a 'dynamical roughness' turns out to be smaller on the hill top, where, as already said, turbulence (and thus eddies activity) is weaker than on the valleys.

We have already pointed out the presence of an upwind-downwind symmetry inside the 'logarithmic layer' for both $u_\star^{\text{eff}}(x)$ and $z_0^{\text{eff}}(x)$. Physically, this means that, sufficiently far above the topography, it is just the mean cumulative result of many almost independent effects, arising in the underlying layer, which is relevant for the large scale dynamics. The

details of the underlying dynamics (e.g., the regime of flow separation) only affects the numerical values of quantities related to the effective parameters (e.g., its average values along the hill and the amplitude of the modulation).

We can finally investigate the relation between the asymptotic and the pre-asymptotic effective parameters. To that end, averaging the logarithmic profile (3) over the hill periodicity box, we obviously obtain again the log-profile (1), but where now $\mathsf{u}_\star^{\rm eff}$ and $\mathsf{z}_0^{\rm eff}$ are related to the effective parameters at smaller scales. Specifically,

$$\mathsf{u}_\star^{\rm eff} = \langle u_\star^{\rm eff}(x)\rangle \qquad\qquad \ln \mathsf{z}_0^{\rm eff} = \frac{\langle u_\star^{\rm eff}(x)\ \ln z_0^{\rm eff}(x)\rangle}{\langle u_\star^{\rm eff}(x)\rangle} \quad , \tag{4}$$

which do not depend on the position along the hill. The results thus remain unchanged when the average is performed over scales larger than the hill periodicity box.
The relationships between the values of these parameters and the topography characteristics are discussed, for instance, in Refs. [2, 6]. It is easily checked that the first of the two above relations is an immediate consequence of the conservation of the total flux of vertical momentum inside the whole domain. More interestingly, the second relation tells us that the roughness parameter $\mathsf{z}_0^{\rm eff}$ does not have solely a geometrical meaning, being in fact related to $u_\star^{\rm eff}(x)$ at smaller scales. This fact places in an unfavorable light attempts to evaluate, either experimentally or numerically, the parameter $\mathsf{z}_0^{\rm eff}$ without taking into account its dependence on the flow configuration as a whole.

We stress that the possibility of describing the flow in terms of effective parameters is a direct consequence of an intrinsic scale separation in the dynamics. There is a simple (and more treatable) physical system where this link between scale separation and effective parameters can be rigorously stated. This is the transport of passive scalar field (e.g., the density of particles or dye injected into the flow). In this problem, if a small-scale turbulent velocity field $\mathbf{v}(\mathbf{x}, t)$ (varying on scales of the order of l_0) is superimposed to an uniform field $\mathbf{V}$, for times large compared with those characteristic of the turbulent field, the concentration field Θ is now varying on scales $L >> l_0$ and, as shown in Ref. [15], obeys the Smoluchowsky equation:

$$\partial_t\Theta + (\mathbf{V}\cdot\partial)\,\Theta = \partial(D\,\partial)\Theta \tag{5}$$

where D is the so-called eddy-diffusivity, i.e. an effective enhanced diffusivity which depends on the characteristics of turbulence as a whole.

Furthermore, under the assumption that $\mathbf{V}$ is now varying on spatial scales of the order of $L >> l_0$, it has been shown (see Ref. [16]) that again a Smoluchowsky equation for Θ holds for large times, but where the effective diffusivity now takes a smooth dependence on the position on scales of the order of L.

The existence of some analogy between the two aforesaid contexts seems clear. The role played by $\mathbf{v}(\mathbf{x}, t)$ in the passive scalar dynamics is played in our problem by the

smallest eddies placed in the vicinity of the wall and produced by the roughness elements (flat surface) or by the topography (flow averaged over distances much larger than the topography wave-length). The statistical role of such small scale features affects the large scale dynamics only via parameters such as D for the passive scalar problem and such as $u_\star$ and z_0 in the case of the flow above a flat surface or $\mathsf{u}_\star^{\rm eff}$ and $\mathsf{z}_0^{\rm eff}$ in the case of the flow over hilly terrain, averaged at a scale much larger than the topography wave-length (asymptotic regime). On the other hand, the passive scalar dynamics observed at the same scales (of the order of L) of the varying advecting velocity, $\mathbf{V}$, corresponds in our problem to the velocity behavior above a hilly terrain, when this velocity field is observed at scales comparable with the integral scale of the problem (the wave-length, λ, of the hill). In such (pre-asymptotic) regimes, the dynamics is again described in terms of effective parameters but now depending on the position, i.e. $u_\star^{\rm eff}(x)$ and $z_0^{\rm eff}(x)$ in the case of flow above a hilly surface, $D(\mathbf{x})$ for the passive scalar problem.

The analogy cannot be pushed any further, but we are convinced that it is a clear indication that the scenario we have described is a direct consequence of an intrinsic scale separation in the dynamics.

In conclusion, we have presented strong numerical evidences showing the 'local' validity of the law-of-the-wall not only above flat terrain, where it is well known, but also in the presence of hilly terrain. The logarithmic shape for the velocity field is restored far enough from the terrain where typical scales of the velocity fields appear "separated" from those relative to the smallest eddies confined near the surface. Local logarithmic profiles are well evident and described in terms of effective parameters showing a smooth dependence on the position along the hill. We have found [17] (but not discussed in this Letter) similar results analyzing data from both wind tunnel experiments (see Ref. [11]) and experiments in nature (see Ref. [12]). Such results confirm the scenario here outlined and also give a strong evidence that our conclusions do not depend on the particular choice of the parameterization scheme adopted in the numerical model here considered.
The relation between asymptotic and pre-asymptotic effective parameters is also derived and the dynamical role of the roughness clearly emerges. Finally, a simple analogy with the pre-asymptotic dynamics for the passive scalar problem, is exploited to confirm our interpretation of the reasons underlying the presented numerical evidence.

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